[Proof] Weighted Jacobi

- 問題

$$\boldsymbol{Au = f}$$

- Jacobi 更新式

$$\boldsymbol{v}^{k+1} = R_J \boldsymbol{v}^k + D^{-1} \boldsymbol{f}$$

$$\begin{aligned}
(D - B)\boldsymbol{u} &= \boldsymbol{f}, \quad (A = D - B) \\
D\boldsymbol{u} - B\boldsymbol{u} &= \boldsymbol{f} \\
D\boldsymbol{u} &= B\boldsymbol{u} + \boldsymbol{f} \\
\boldsymbol{u} &= D^{-1}B\boldsymbol{u} + D^{-1}\boldsymbol{f} \\
\boldsymbol{u} &= R_J \boldsymbol{u} + D^{-1}\boldsymbol{f}
\end{aligned}$$

- 重み ω を導入

$$\omega \in [0, 1]$$

- Weighted Jacobi 更新式

$$\boldsymbol{v}^{k+1} = R_J \boldsymbol{v}^k + \omega D^{-1} \boldsymbol{f}$$

$$\begin{aligned}
\boldsymbol{u} &= (1 - \omega)\boldsymbol{u} + \omega\boldsymbol{u} \\
&= (1 - \omega)\boldsymbol{u} + \omega\left(R_J\boldsymbol{u} + D^{-1}f\right) \\
&= ((1 - \omega)I + \omega R_J)\,\boldsymbol{u} + \omega D^{-1}\boldsymbol{f} \\
&= R_\omega \boldsymbol{u} + \omega D^{-1}\boldsymbol{f}
\end{aligned}$$

[Proof] Weighted Jacobi

- 残差

$$\boxed{\boldsymbol{r} = \boldsymbol{f} - A\boldsymbol{u}}$$

- 残差を使って表現

$$\boxed{\boldsymbol{v}^{k+1} = \boldsymbol{v}^k + \omega D^{-1}\boldsymbol{r}^k}$$

- $R_J = D^{-1}B$
- $R_\omega = (1-\omega)I + \omega R_J$

$$
\begin{aligned}
\boldsymbol{v}^{k+1} &= R_\omega \boldsymbol{v}^k + \omega D^{-1}\boldsymbol{f} \\
&= R_\omega \boldsymbol{v}^k + \omega D^{-1}\left(\boldsymbol{r}^k + A\boldsymbol{v}^k\right) \\
&= R_\omega \boldsymbol{v}^k + \omega D^{-1}\left(\boldsymbol{r}^k + (D-B)\boldsymbol{v}^k\right) \\
&= R_\omega \boldsymbol{v}^k + \omega D^{-1}\boldsymbol{r}^k + \omega\boldsymbol{v}^k - \omega D^{-1}B\boldsymbol{v}^k \\
&= ((1-\omega)I + \omega R_J)\boldsymbol{v}^k \\
&\quad + \omega D^{-1}\boldsymbol{r}^k + \omega\boldsymbol{v}^k - \omega R_J\boldsymbol{v}^k \\
&= \boldsymbol{v}^k \cancel{-\omega\boldsymbol{v}^k} + \cancel{\omega R_J\boldsymbol{v}^k} \\
&\quad + \omega D^{-1}\boldsymbol{r}^k + \cancel{\omega\boldsymbol{v}^k} + \cancel{R_J\boldsymbol{v}^k} \\
&= \boldsymbol{v}^k + \omega D^{-1}\boldsymbol{r}^k
\end{aligned}
$$

[Proof] Weighted Jacobi

- $e = u - v^k$
- $r^k = f - Av^k$, $Av^k = f - r^k$
- $r - r^k = -Ae^k$

$$
\begin{aligned}
r - r^k &= \cancel{f} - Au - \left(\cancel{f} - Av^k\right) \\
&= -A(u - v^k) \\
&= -A(e)
\end{aligned}
$$

$$u = u + \omega D^{-1} r \tag{1}$$

$$v^{k+1} = v^k + \omega D^{-1} r^k \tag{2}$$

$$
\begin{aligned}
(1) - (2) \qquad u - v^{k+1} &= u + \omega D^{-1} r - v^k - \omega D^{-1} r^k \\
u - v^{k+1} &= u - v^k + \omega D^{-1}\left(r - r^k\right) \\
e^{k+1} &= e^k + \omega D^{-1} * -Ae^k
\end{aligned}
$$

$$e^{k+1} = \left(I - \omega D^{-1} A\right) e^k \tag{3}$$

[Proof] Weighted Jacobi

- $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{v}^k$
- $\boldsymbol{r}^k = \boldsymbol{f} - A\boldsymbol{v}^k$, $A\boldsymbol{v}^k = \boldsymbol{f} - \boldsymbol{r}^k$
- $\boldsymbol{r} - \boldsymbol{r}^k = -A\boldsymbol{e}^k$

- $\boldsymbol{e}^{k+1} \approx \boldsymbol{e}^k$

緩和をしても誤差が減らない

$$
\begin{aligned}
\boldsymbol{e}^{k+1} &= \left(I - \omega D^{-1}A\right)\boldsymbol{e}^k \\
\boldsymbol{e}^{k+1} &= \boldsymbol{e}^k - \omega D^{-1}A\boldsymbol{e}^k \\
\boldsymbol{e}^{k+1} - \boldsymbol{e}^k &= -\omega D^{-1}A\boldsymbol{e}^k \\
\boldsymbol{0} &\approx -\omega D^{-1}A\boldsymbol{e}^k
\end{aligned}
$$

$$A\boldsymbol{e} \approx \boldsymbol{0}$$

$$
\begin{aligned}
A\boldsymbol{e}^k &= A(\boldsymbol{u} - \boldsymbol{v}^k) \approx \boldsymbol{0} \\
&= A\boldsymbol{u} - A\boldsymbol{v}^k \approx \boldsymbol{0} \\
&= A\boldsymbol{u} - (\boldsymbol{f} - \boldsymbol{r}^k) \approx \boldsymbol{0} \\
&= \cancel{A\boldsymbol{u} - \boldsymbol{f}} + \boldsymbol{r}^k \approx \boldsymbol{0}
\end{aligned}
$$

$$\boldsymbol{r}^k \approx \boldsymbol{0}$$